NOTES

Gyroscope Unbalance Torques Resulting from Static Linear Compliances

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This paper is devoted to the derivation of the equations for unbalance torque of a single-degree-of-freedom gyroscope due to static linear compliances of the gyroscope under the forces of gravity and acceleration. The theory of classical elasticity is followed so that the analysis is valid for applied force inputs with frequency spectra sufficiently below the natural frequency of elastic vibration of the gyroscope. It is assumed that an applied force produces displacements in the direction of the force as well as in the directions normal to the force. These latter displacements have not been considered before and arise from the fact that compliance transforms like a rectangular Cartesian tensor that is not, in general, symmetric. Unbalance torques due to compliance effects are derived for three typical cases. The results derived are useful for analysis of gyroscope test data in an attempt to improve compensation for elastic deflections of the center of gravity of the gyroscope. This provides a possible way of identifying and evaluating cross compliances by means of appropriate tests.

I. Introduction

EQUATIONS are derived relating unbalance torques of a single-degree-of-freedom gyroscope to the elastic deflections of the center of gravity of the gyroscope associated with the forces of gravity and acceleration. The derivation does not take account of dynamic effects, i.e., it is assumed that the elastic deflections are a function only of the forces and not of the time-history of the situation. Thus the analysis is valid for force inputs with frequency spectra sufficiently below the natural frequency of elastic vibration of the gyro structure.

Equations are derived in a form suitable for application to the analysis of gyro test data; for example, one case covered is that of data derived from tests where, by mounting with suitable degrees of freedom, the gyro is made to hold its orientation in inertial space.[2]

Previous work in the study of compliance has assumed the existence of a set of orthogonal principal axes, so that forces along these axes produce displacements only in the direction of the force. Here it is assumed that displacements normal to the force also can occur, no matter what the direction of the applied force.

II. Derivation of Equations

Let the input, output, and spin reference axes of the gyro form an orthogonal coordinate system as in Fig. 1. Henceforward these axes will be labeled *IA*, *OA*, and *SRA*, respectively. Then if a force **F** is applied, the deflection of the center of gravity resulting from elastic compliance of the structure can be found by resolving **F** into components along the orthogonal axes and superposing the effects of each component. Following classical elasticity theory, assume that compliance deflections are linear with force. Now define a set of compliance components according to the direction of the deflection and the direction of the applied force. In Table 1, the first subscript indicates the direction of the deflection, and the second indicates the applied force component.

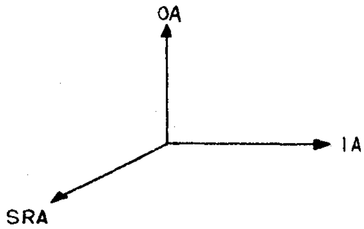

Fig. 1

Compliance as just defined transforms like a rectangular Cartesian tensor and thus will possess a set of mutually perpendicular principal axes if, and only if, the tensor is symmetric, i.e., $K_{SI} = K_{IS}$, $K_{IO} = K_{OI}$, and $K_{SO} = K_{OS}$.[1] In this case, along the principal axes the cross compliances are zero and the transformed matrix reduces to a diagonal matrix. When this condition is fulfilled and one of the principal axes coincides with *OA*, the form of the unbalance torque has been derived. In this development the complete generality eliminates the need for assuming the existence of principal axes.

In general, compliance has been measured by tests in which the input force, commonly gravity, is rotated with respect to the gyro structure, and the resulting torques are subjected to a harmonic analysis. In what follows, unbalance torque due to compliance effects will be derived for three typical cases. If K is the compliance matrix just defined, **F** the input force, and $\mathbf{\Delta}_{e1}$ the elastic deflection resulting, both resolved along the gyro axes, the deflection relationship is

$$\mathbf{\Delta}_{e1} = K\mathbf{F} \tag{1}$$

or

$$\begin{aligned} \Delta_{SRA} &= K_{SS}F_S + K_{SI}F_I + K_{SO}F_O \\ \Delta_{IA} &= K_{IS}F_S + K_{II}F_I + K_{IO}F_O \\ \Delta_{OA} &= K_{OS}F_S + K_{OI}F_I + K_{OO}F_O \end{aligned} \tag{1a}$$

The torque resulting from the deflection $\mathbf{\Delta}$ and specific force **F** is then

$$\mathbf{T} = \mathbf{\Delta} \times \mathbf{F} \tag{2}$$

with a component along *OA* equal to

$$T_{OA} = \mathbf{\Delta} \times \mathbf{F} \cdot \mathbf{1}_{OA} \tag{2a}$$

Case 1: Input force makes a fixed angle with OA (see Fig. 2)

Here *OA* is parallel to the earth's axis (*EA*), and the gyro is rotated about *OA*. Let β be the angle between **g** and the *IA-SRA* plane, and let ϕ be the angle from *SRA* to the projection of **g** in the *IA-SRA* plane. Then the components of **g** are

$$\begin{aligned} g_{OA} &= -g\sin\beta \\ g_{SRA} &= -g\cos\beta\cos\phi \\ g_{IA} &= -g\cos\beta\sin\phi \end{aligned} \tag{3}$$

Let Δ_{SRA}, Δ_{IA}, and Δ_{OA} be the components of elastic deflection along the three axes and M the mass. For a single-degree-of-freedom gyroscope, Δ_{OA} will make no contribution to torque about the output axis and can be neglected. Now

$$\begin{aligned} \Delta_{SRA} &= -Mg(K_{SS}\cos\beta\cos\phi + K_{SI}\cos\beta\sin\phi + K_{SO}\sin\beta) \\ \Delta_{IA} &= -Mg(K_{IS}\cos\beta\cos\phi + K_{II}\cos\beta\sin\phi + K_{IO}\sin\beta) \\ \Delta_{OA} &= -Mg(K_{OS}\cos\beta\cos\phi + K_{OI}\cos\beta\sin\phi + K_{OO}\sin\beta) \end{aligned} \tag{4}$$

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Table 1 Definition of compliance components

Compliance components	Deflection of center of gravity along	Due to unit force along
K_{SS}	SRA	SRA
K_{SI}	SRA	IA
K_{SO}	SRA	OA
K_{IS}	IA	SRA
K_{II}	IA	IA
K_{IO}	IA	OA
K_{OS}	OA	SRA
K_{OI}	OA	IA
K_{OO}	OA	OA

In addition, assume that there is a constant torque R about OA produced by another source and that there are static displacements (U_{SRA} and U_{IA}) of the center of gravity along SRA and IA from the center of support. Then the torque about OA produced by these effects and the elastic displacement will be

$$T_{OA} = R + Mg\,U_{SRA}\cos\beta\sin\phi - Mg\,U_{IA}\cos\beta\cos\phi - \Delta_{SRA}\,Mg_{IA} + \Delta_{IA}\,Mg_{SRA} \quad (5)$$

Substitution of Eq. (4) into (5) and combining terms produces

$$T_{OA} = [R + \tfrac{1}{2}M^2g^2(K_{IS} - K_{SI})\cos^2\beta] + [Mg\,U_{SRA}\cos\beta - \tfrac{1}{2}M^2g^2K_{SO}\sin2\beta]\sin\phi + [-Mg\,U_{IA}\cos\beta + \tfrac{1}{2}M^2g^2K_{IO}\sin2\beta]\cos\phi + \tfrac{1}{2}M^2g^2(K_{II} - K_{SS})\cos^2\beta\sin2\phi + \tfrac{1}{2}M^2g^2(K_{IS} - K_{SI})\cos^2\beta\cos2\phi \quad (6)$$

Case 2:* *Input force makes a fixed angle with IA (see Fig. 3)

This case differs from case 1 in the orientation of the gyroscope and in the assumption that the gyroscope is rotated about IA, which is parallel to EA. The equations corresponding to Eqs. (3–6) of the preceding section become

$$g_{IA} = -g\sin\beta \qquad g_{SRA} = -g\cos\beta\cos\phi \quad (7)$$

$$g_{OA} = g\cos\beta\sin\phi$$

$$\Delta_{SRA} = Mg(-K_{SS}\cos\beta\cos\phi - K_{SI}\sin\beta + K_{SO}\cos\beta\sin\phi)$$
$$\Delta_{IA} = Mg(-K_{IS}\cos\beta\cos\phi - K_{II}\sin\beta - K_{IO}\cos\beta\sin\phi) \quad (8)$$
$$\Delta_{OA} = Mg(-K_{OS}\cos\beta\cos\phi - K_{OI}\sin\beta - K_{OO}\cos\beta\sin\phi)$$

$$T_{OA} = R + Mg\,U_{SRA}\sin\beta - Mg\,U_{IA}\cos\beta\cos\phi - \Delta_{SRA}\,Mg_{IA} + \Delta_{IA}\,Mg_{SRA} \quad (9)$$

$$T_{OA} = (R + Mg\,U_{SRA}\sin\beta + \tfrac{1}{2}M^2g^2\,K_{IS}\cos^2\beta - M^2g^2\,K_{SI}\sin^2\beta) + \tfrac{1}{2}M^2g^2\,K_{SO}\sin2\beta\sin\phi + [-Mg\,U_{IA}\cos\beta + \tfrac{1}{2}M^2g^2(K_{II} - K_{SS})\sin2\beta]\cos\phi - \tfrac{1}{2}M^2g^2K_{IO}\cos^2\beta\sin2\phi + \tfrac{1}{2}M^2g^2K_{IS}\cos^2\beta\cos2\phi \quad (10)$$

Terms here have been collected according to the harmonics of ϕ, which is the angle of rotation of the test table on which the gyroscope is mounted. When the gyroscope wheel is not rotating, the tests can be performed at a more or less arbitrary table rotation rate. When the gyroscope wheel is rotating, the motion of the base in inertial space can produce torques about the output axis, and these must be taken into account. Earth's rotation may be removed by rotating the table in the opposite direction at one earth rate so that the gyroscope preserves its orientation in inertial space, using an independent source for the time drive. In this case the angle ϕ goes through a range of 2π rad in one sidereal day. An alternative method applicable to case 2 is to drive the table by an angular velocity signal from the gyro signal generator, instead of using an independent time drive.

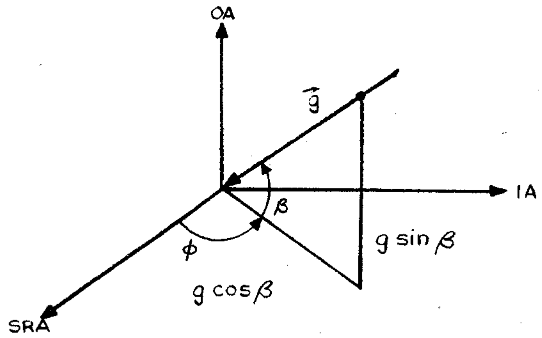

Fig. 2

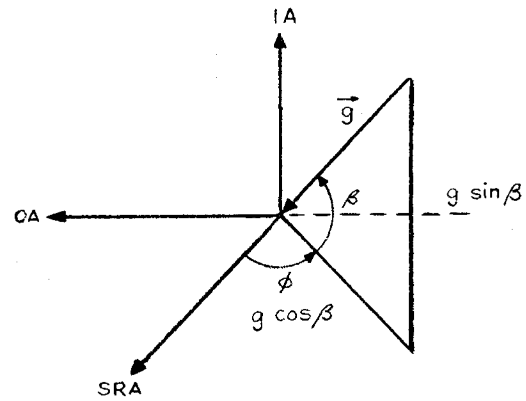

Fig. 3

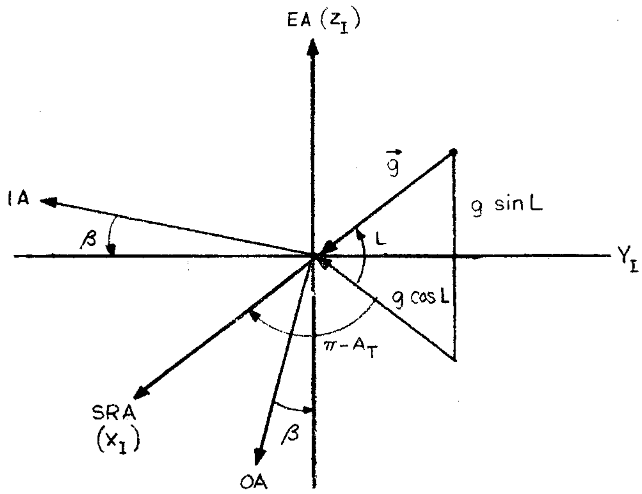

Fig. 4

Case 3:* *SRA fixed in the equatorial plane (see Fig. 4)

Let the test table axis be parallel to EA, and assume that a sidereal time drive is applied to remove earth's rotation, so that the gyroscope preserves its orientation in inertial space. Let $SRA = X_I$ be normal to $EA = Z_I$, and let Y_I complete the orthogonal system. Then the $X_I\,Y_I$ plane is parallel to the equatorial plane of the earth. Let A_T be the angle from the projection of $\mathbf{g}$ into the $X_I\,Y_I$ plane to SRA (positive direction counterclockwise about EA), and let β be the angle from IA to the $X_I\,Y_I$ plane and from OA to $-EA$ (positive direction counterclockwise about SRA).

Then the components of $\mathbf{g}$ are

$$g_{SRA} = g\cos L\cos A_T$$
$$g_{IA} = g\cos L\sin A_T\cos\beta - g\sin L\sin\beta \quad (11)$$
$$g_{OA} = g\cos L\sin A_T\sin\beta + g\sin L\cos\beta$$

Proceeding as before, one obtains the components of $\mathbf{\Delta}$:

$$\begin{aligned}\Delta_{SRA} &= U_{SRA} + Mg\,K_{SS}\cos L\cos A_T + \\ &\quad Mg\,K_{SI}(\cos L\sin A_T\cos\beta - \sin L\sin\beta) + \\ &\quad Mg\,K_{SO}(\cos L\sin A_T\sin\beta + \sin L\cos\beta) \\ \Delta_{IA} &= U_{IA} + Mg\,K_{IS}\cos L\cos A_T + \\ &\quad Mg\,K_{II}(\cos L\sin A_T\cos\beta - \sin L\sin\beta) + \\ &\quad Mg\,K_{IO}(\cos L\sin A_T\sin\beta + \sin L\cos\beta)\end{aligned} \tag{12}$$

From the preceding, the unbalance torque about OA is derived:

$$\begin{aligned}T_{OA} &= R + \Delta_{SRA}F_{IA} - \Delta_{IA}F_{SRA} = \\ &R - Mg\,U_{SRA}\sin L\sin\beta - \tfrac{1}{2}M^2g^2K_{IS}\cos^2 L + \\ &\tfrac{1}{4}M^2g^2K_{SO}(\cos^2 L - 2\sin^2 L)\sin 2\beta + \\ &\tfrac{1}{2}M^2g^2K_{SI}(\cos^2 L\cos^2\beta + 2\sin^2 L\sin^2\beta) + \\ &[Mg\,U_{SRA}\cos L\cos\beta - \tfrac{1}{4}M^2g^2K_{SI}\sin 2L\sin 2\beta + \\ &\tfrac{1}{2}M^2g^2K_{SO}\sin 2L\cos 2\beta]\sin A_T + \\ &[-Mg\,U_{IA}\cos L + \tfrac{1}{2}M^2g^2(K_{II} - K_{SS})\sin 2L\sin\beta - \\ &\tfrac{1}{2}M^2g^2K_{IO}\sin 2L\cos\beta]\cos A_T - \\ &[\tfrac{1}{2}M^2g^2(K_{II} - K_{SS})\cos^2 L\cos\beta + \\ &\tfrac{1}{2}M^2g^2K_{IO}\cos^2 L\sin\beta]\sin 2A_T - \\ &[\tfrac{1}{2}M^2g^2K_{SI}\cos^2 L\cos^2\beta + \tfrac{1}{2}M^2g^2K_{IS}\cos^2 L + \\ &\tfrac{1}{2}M^2g^2K_{SO}\cos^2 L\sin 2\beta]\cos 2A_T\end{aligned} \tag{13}$$

Once again the torque has been expressed in harmonics of A_T.

III. Summary

The results derived in the preceding section are useful for analysis of gyroscope test data in an attempt to improve compensation for elastic deflections. By means of appropriate tests, this provides a possible way of identifying and evaluating cross compliances. For example, a set of tests may be made with different values of β in case 1. For each value of β, a harmonic analysis will provide a coefficient of each harmonic of ϕ. Then the coefficients themselves may be analyzed by harmonic analysis in terms of β to evaluate the cross compliances. The accuracy of the analysis will depend strongly on the amount and nature of the interference or "noise" in the original data.

References

[1] Sokolnikoff, I. S., *Tensor Analysis: Theory and Applications* (John Wiley and Sons Inc., New York, 1951), pp. 34–46.

[2] Denhard, W. T., "Laboratory testing of a floated single degree of freedom integrating gyro," Mass. Inst. Tech. Instrumentation Lab. Rept. R-105.

Energy Separation in Laminar Vortex-Type Slip Flow

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Nomenclature

- c_p = specific heat at constant pressure
- d = tube diameter, $2r_0$
- k = thermal conductivity
- p = static pressure
- Pr = Prandtl number, $\mu c_p/k$
- q = heat transfer to the wall
- r = radial coordinate for tube; r_0 is tube radius
- T = temperature; T_w is wall temperature; T_g is gas temperature adjacent to wall; T_t is total temperature; $T_{t,g}$ is total temperature of gas adjacent to wall
- u = axial velocity; $\bar{u}$ is average velocity; u_c is centerline velocity; u_s is slip velocity
- v = tangential velocity; v_0 is velocity at radius r_0; v_s is slip velocity
- x = axial coordinate for tube
- η = dimensionless coordinate, r/r_0
- λ = dimensionless velocity, $u_s/\bar{u}$
- μ = absolute viscosity
- ξ_v = velocity slip coefficient
- ρ = density

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This note is concerned with the effects of low-density phenomena on energy separation for laminar fluid flow in an insulated, rotating circular tube. Specifically, consideration is given to the slip-flow regime, wherein velocity and temperature discontinuities occur at the tube wall. It is felt that this study will provide some insight into the energy separation characteristics of a vortex tube operating under slip-flow conditions. Incompressible flow is assumed; in addition, it is assumed that the velocity-slip coefficient does not depend on the direction of the flow with respect to the surface, so that the single coefficient ξ_v suffices.

The appropriate forms of the Navier-Stokes equations (assumed to govern the flow) are

$$(\mu/r)(\partial/\partial r)[r(\partial u/\partial r)] = \partial p/\partial x \tag{1}$$

$$(\partial/\partial r)[(1/r)(\partial/\partial r)(rv)] = 0 \tag{2}$$

The boundary conditions imposed on u and v are[1]

$$\begin{aligned} &u \equiv u_s \equiv -\xi_v(du/dr)_{r=r_0} && \text{at } r = r_0 \\ &du/dr = 0 && \text{at } r = 0 \\ &v = v_0 - v_s \equiv v_0 - \xi_v(dv/dr)_{r=r_0} && \text{at } r = r_0 \\ &v = 0 && \text{at } r = 0 \end{aligned} \tag{3}$$

The velocity components as obtained by solution of these equations are

$$\begin{aligned} u &= 2\bar{u}[1 - \eta^2 + 4(\xi_v/d)]/[1 + 8(\xi_v/d)] \\ v &= v_0\eta/[1 + 2(\xi_v/d)] \end{aligned} \tag{4}$$

where $\eta \equiv r/r_0$. The energy equation for this case is

$$\rho c_p u(\partial T/\partial x) = (kr)(\partial/\partial r)[r\partial T/\partial r] + \mu(du/dr)^2 \tag{5}$$

subject to the boundary condition[2]

$$q = [k(\partial T/\partial r) + \mu u(\partial u/\partial r)]_{r=r_0} \equiv 0 \quad \text{(insulated wall)} \tag{6}$$

(The tangential flow produces no additional shear stresses.)

Equations (5) and (6) may be rewritten in the form[3]

$$\begin{aligned}\rho c_p u(\partial/\partial x)[T + (Pr/c_p)(u^2/2)] &= (8\mu\bar{u}/r_0^2)(1 - \lambda)u + \\ &(kr)(\partial/\partial r)\{r(\partial/\partial r)[T + (Pr/c_p)(u^2/2)]\}\end{aligned} \tag{7}$$

$$q = k(\partial/\partial r)[T + (Pr/c_p)(u^2/2)]_{r=r_0} \equiv 0 \tag{8}$$

Thus, at any given axial location, Eqs. (7) and (8) are satisfied by the solution

$$T + (Pr/c_p)(u^2/2) \equiv \text{const} \equiv T_g + (Pr/c_p)(u_s^2/2) \tag{9}$$

If the static temperature T is converted to the total temperature $[T_t \equiv T + (u^2 + v^2)/2c_p]$ and is made dimensionless through the use of the centerline axial velocity u_c, one obtains, after algebraic manipulation, the following expression:

$$\begin{aligned}(T_t - T_{t,g})/(u_c^2/2c_p) &= (1 - Pr) \times \\ &\{[(2 - \lambda) - 2(1 - \lambda)\eta^2]^2 - \lambda^2\}/(2 - \lambda)^2 - \\ &16(1 - \lambda)^2(v_0/u_c)^2(1 - \eta^2)/(4 - 3\lambda)^2\end{aligned} \tag{10}$$

The result is shown in Fig. 1 for several values of the parameters (v_0/u_c) and $(u_s/\bar{u})$ and for a Prandtl number of 0.7.

Inspection of this figure reveals the effect of the rarefaction. With continuum flow ($\lambda = 0$), a considerable variation in total temperature is realized for Poiseuille flow through a rotating tube, the effect becoming larger with in-